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On the construction of a complete fan associated to the moduli space of polarized logarithmic Hodge structures

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1 Introduction

The notion of polarized logarithmic Hodge structure (PLH) was introduced by professor Kazuya Kato and professor Sampei Usui. Roughly speaking, the ne moduli space of PLH is described as follows.

\D=(polarized Hodge structures with a " -level structure ")
∩
\D = \{ -nilpotent orbit | ∈ }
=(polarized logarithmic Hodge structures with a " -level structure "whose "local monodromies are in the directions in ")

Here D is a classifying space of polarized Hodge structures, is a discrete subgroup of Aut(D), and is a fan consisting of rational nilpotent cones in Lie(Aut(D)) which is strongly compatible with .

In the classical situation, that is, D is a symmetric Hermitian domain, the toroidal projective compacti cation \D of \D was constructed with a su ciently big fan , called a projective fan, by A. Ash, D. Mumford, M. Rapoport and Y. S. Tai.

For general D, Kato and Usui introduced a "complete fan "as a generalization of a projective fan, and they gave a conjecture of the existence of such fans.

Example (KU) For D with $h^{2,0} = h^{0,2} = 2$, $h^{1,1} = 1$, the fan Ξ consisting of all rational nilpotent cones whose rank are less than or equal to one in Lie(Aut(D)) is complete.

Theorem There is no complete fan for D with $h^{2,0} = h^{1,1} = h^{0,2} = 2$.

By Theorem, the de nition of complete fan was modi ed by Chikara Nakayama. Now, I try to construct a complete fan in the new sense for D with $h^{2,0} = h^{1,1} = h^{0,2} = 2$.

2 Nilpotent orbit

In this section, we recall the de nition of nilpotent orbit. We x a 4-tuple $\Phi_0 = (w, (h^{p,q})_{p,q\in\mathbb{Z}}, H_{\mathbb{Z}}, \langle\ ,\ \rangle)$, where $H_{\mathbb{Z}}$ is a free $\mathbb{Z}$-module of rank $\ _{p,q}h^{p,q}$, and $\langle\ ,\ \rangle$ is a non-degenerate bilinear form on $H_{\mathbb{Q}} := \mathbb{Q}\otimes_{\mathbb{Z}} H_{\mathbb{Z}}$ which is symmetric if w is even and skew-symmetric if w is odd. Let D be the classifying space of polarized Hodge structures of type Φ_0, and $\check{D}$ be the compact dual of D.

Let

$$G_{\mathbb{Z}} := \mathrm{Aut}(H_{\mathbb{Z}}, \langle\ ,\ \rangle)$$

and for $R = \mathbb{Q}, \mathbb{R}, \mathbb{C}$, let $H_R := R\otimes_{\mathbb{Z}} H_{\mathbb{Z}}$, $G_R := \mathrm{Aut}(H_R, \langle\ ,\ \rangle)$
$\mathfrak{g}_R := \mathrm{Lie}(G_R)$
$= \{N \in \mathrm{End}_R(H_R) \mid \langle Nx, y\rangle + \langle x, Ny\rangle = 0$ for all $x, y \in H_R\}$.

Definition (KU) A subset of $\mathfrak{g}_{\mathbb{R}}$ is said to be a nilpotent cone, if the following conditions are satis ed.

(1) $= \mathbb{R}_{\geq 0}N_1 + \cdots + \mathbb{R}_{\geq 0}N_n$ for some $n \geq 1$ and for some $N_1, \dots, N_n \in$.

(2) Any element of is nilpotent as an endomorphism of $H_{\mathbb{R}}$.

(3) $[N, N'] = 0$ for any $N,\ N' \in$ as endomorphisms of $H_{\mathbb{R}}$, where $[N, N'] := NN' - N'N$. A nilpotent cone is said *rational*, if we can take $N_1, \dots, N_n \in \mathfrak{g}_{\mathbb{Q}}$

Definition (KU) Let $= \ _{1\leq j\leq r}(\mathbb{R}_{\geq 0})N_j$ be a rational nilpotent cone. A subset Z of $\check{D}$ is said to be a -nilpotent orbit if there is $F \in \check{D}$ which satis es $Z = \exp(\ _{\mathbb{C}})F$ and satis es the following two conditions.

(1) $N_jF^p \subset F^{p\ \ 1}$ $(1 \leq j \leq r, p \in \mathbb{Z})$.

(2) $\exp(\sum_{1\leq j\leq r} z_jN_j)F \in D$ if $z_j \in \mathbb{C}$ and $\mathrm{Im}(z_j) \gg 0$.

3 Complete fan

Let be a fan in $\mathfrak{g}_{\mathbb{Q}}$, i.e., a fan consisting of rational nilpotent cones.

Definition (N) is complete in the new sense if it satis es the following condition.

$$\bigcup_{\sigma\in} = \bigcup_{\sigma\in \mathrm{N}}$$

, where N is the set of all rational nilpotent cones in $\mathfrak{g}_{\mathbb{R}}$ such that there is a nilpotent orbit.

4 A special case

In this section, we consider the construction of a complete fan in the new sense for D with $h^{2,0} = h^{1,1} = h^{0,2} = 2$. Let $H_{\mathbb{Z}} = \bigoplus_{1\leq j\leq 6}\mathbb{Z}e_j$, and let $(\langle e_i, e_j\rangle)_{1\leq i,j\leq 6} = \begin{pmatrix} -1_2 & O & O \\ O & H & O \\ O & O & H \end{pmatrix}$, where $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Let $H''_{\mathbb{Q}} = \mathbb{Q}e_1 \oplus \mathbb{Q}e_2$, and let $a \in H''_{\mathbb{Q}}$. De ne $N_a, N'_a, N_0 \in \mathfrak{g}_{\mathbb{Q}}$ as follows.

$$N_a(e_i) = -\langle a, e_i\rangle e_5\ (i = 1, 2),\ N_a(e_6) = a,$$
$$N_a(e_i) = 0\ (i \neq 1, 2, 6);$$
$$N'_a(e_i) = -\langle a, e_i\rangle e_3\ (i = 1, 2),\ N'_a(e_4) = a,$$
$$N'_a(e_i) = 0\ (i \neq 1, 2, 4);$$
$$N_0(e_4) = -e_5,\ N_0(e_6) = e_3,\ N_0(e_i) = 0\ (i \neq 4, 6).$$

Then, for $a, b, c \in H''_{\mathbb{Q}}\backslash\{0\}$ such that $\langle a, b\rangle = 0$, we consider the rational nilpotent cones of rank one $(a, b) = \mathbb{R}_{\geq 0}(N_a + N'_b)$, $(c) = \mathbb{R}_{\geq 0}N_c$, and $\ _0 = \mathbb{R}_{\geq 0}N_0$. We note that these nilpotent cones are belong to N. Denote the weight ltration associated to (a, b), (c), and to $\ _0$ by W^1, W^2 and W^3 respectively.

Here, let $G^0_{\mathbb{R}}$ be the connected component of $G_{\mathbb{R}}$ containing 1, for the Zariski topology. For each i, let P_i be the $\mathbb{Q}$-parabolic subgroup of $G_{\mathbb{R}}$ consisting of all elements g of $G^0_{\mathbb{R}}$ such that $gW^i = W^i$. Let $\mathfrak{p}_i$ be the Lie algebra of P_i, let $n(\mathfrak{p}_i)$ be the nilradical of $\mathfrak{p}_i$, and let N_i be the set of all elements $\in$ N such that $\subset n(\mathfrak{p}_i)$.

In general, for a rational nilpotent cone , denote the weight ltration associated to by $W(\ )$. Since, for any $\in$ N, there exists an element $g \in G_{\mathbb{Q}}$ and $i \in \{1, 2, 3\}$ such that $gW(\ ) = W^i$, and since, if $W(\ ) = W^i$ for some i, then $\in \mathrm{N}_i$, we try to construct a complete fan in the new sense which is compatible with $G_{\mathbb{Z}}$ by the following ways.

Method 1. We try to construct a "local complete fan $\ _0$ "satisfying $\bigcup_{\sigma\in\ _0} = \bigcup_{\sigma\in \mathrm{N}_0}$. Here $\mathrm{N}_0 = \bigcup_{1\leq i\leq 3}\mathrm{N}_i$.

2. We try to glue the fans obtained by the adjoint actions of all elements of $G_{\mathbb{Z}}$ on $\ _0$.